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ORIGINAL PAPER

Binomial partial Steiner triple systems containing complete graphs

Małgorzata Prażmowska[1] · **Krzysztof Prażmowski**[2]

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Abstract We propose a new approach to studies on partial Steiner triple systems consisting in determining complete graphs contained in them. We establish the structure which complete graphs yield in a minimal PSTS that contains them. As a by-product we introduce the notion of a binomial PSTS as a configuration with parameters of a minimal PSTS with a complete subgraph. A representation of binomial PSTS with at least a given number of its maximal complete subgraphs is given in terms of systems of perspectives. Finally, we prove that for each admissible integer there is a binomial PSTS with this number of maximal complete subgraphs.

Keywords Binomial configuration · Generalized Desargues configuration · Complete graph

Mathematics Subject Classification 05B30 · 05C51 · 05B40

1 Introduction

In the paper we investigate the structure which (may) yield complete graphs contained in a (partial) Steiner triple system (in short: in a PSTS). Our problem is, in fact, a particular instance of a general question, investigated in the literature, which STS's

✉ Małgorzata Prażmowska
malgpraz@math.uwb.edu.pl

Krzysztof Prażmowski
krzypraz@math.uwb.edu.pl

1 Faculty of Mathematics and Informatics, University of Białystok, ul. Ciołkowskiego 1M, 15-245 Białystok, Poland

2 Institute of Mathematics, University of Białystok, ul. Ciołkowskiego 1M, 15-245 Białystok, Poland

(more generally: which PSTS's) contain/do not contain a configuration of a prescribed type. Here belongs, e.g. the problem to determine Pasch-free configurations in some definite classes of (P)STS's [13], or to characterize STS's which do not contain the mitre configuration (e.g.: [7,23]). It resembles also (and, in a sense, generalizes) the problem to determine triangle-free (e.g. [1,10]), quadrangle-free (e.g. [3]), and other $\mathcal{K}$-free (e.g. [6,16]) configurations, where $\mathcal{K}$ is a fixed class of configurations (of a special importance). In our case configurations in question are complete graphs and so-called binomial configurations.

In essence, in what follows, speaking about a (complete) graph G contained in a configuration $\mathfrak{M}$ we always assume that G is *freely* contained in $\mathfrak{M}$ i.e. it is not merely a subgraph of the collinearity (point-adjacency) graph of $\mathfrak{M}$ but also distinct edges of G lie on distinct lines (so called *sides*) of $\mathfrak{M}$, and sides do not intersect outside G. Note that following this terminology a complete quadrangle on a projective plane $\mathfrak{P}$ *is not* a K_4-graph *contained* in $\mathfrak{P}$.

There are two main problems that this theory starts with. Firstly: *what are the minimal parameters of a* PSTS *necessary to contain a K_n-graph* and *does there exist a* PSTS *with these parameters which contains/does not contain a K_n-graph*. It turns out that a minimal (with respect to its size) PSTS that contains K_n is a so called *binomial configuration* i.e. a $(v_r\ b_3)$-configuration such that $v = \binom{m}{2}$, $r = m-2$, and $b = \binom{m}{3}$ for a positive integer m $(=n+1)$ (Prop. 3.2). We say, in short, that this PSTS is *a minimal configuration* which contains K_n.

Several classes of binomial configurations were introduced and studied in the literature (see [19, combinatorial Grassmannians], [17, combinatorial Veronesians], [18, multiveblen configurations], [20,21]). The above characterization of the parameters of a binomial configuration gives a good motivation and justification for investigating this class from 'a general perspective'. In most of the binomial configurations already defined in the literature a suitable complete graph can be found. As the best known example of a binomial configuration we can quote generalized Desargues configuration (see [4,5,12]). Then the answer to the 'dual' question *what is the maximal size of a complete graph that a binomial* $\left(\binom{n}{2}_{n-2}\ \binom{n}{3}_3\right)$*-configuration* $\mathfrak{M}$ *may contain* easily follows: this size is $n-1$. In this case we say that K_n is a maximal (complete) subgraph of $\mathfrak{M}$. However, there are $\left(\binom{n}{2}_{n-2}\ \binom{n}{3}_3\right)$-configurations that do not contain any K_{n-1}-graph. The simplest example can be found for $n=5$: it is known that there are 10_3-configurations without K_4. (cf. [2,15]).

The second problem is read as follows: provided a $\left(\binom{n+1}{2}_{n-1}\ \binom{n+1}{3}_3\right)$-configuration contains a K_n, *what is the possible number of K_n-graphs contained in $\mathfrak{M}$ and what is the structure they yield*. It turns out that the maximal number of such K_n-subgraphs is $n+1$ (Prop. 3.9). Binomial configurations with the maximal number of K_n-subgraphs are exactly the generalized Desargues configurations. This fact points out once more a special position of the class of generalized Desargues configurations within the class of binomial configurations.

As we already mentioned, the problem whether a given binomial $\left(\binom{n+1}{2}_{n-1}\ \binom{n+1}{3}_3\right)$-configuration $\mathfrak{M}$ contains a complete K_n-graph is slightly similar to the problem if a PSTS contains a Pasch configuration. Indeed, $\mathfrak{M}$ contains such a graph iff it contains a binomial $\left(\binom{n}{2}_{n-2}\ \binom{n}{3}_3\right)$-configuration (Prop. 3.3, Cor.3.4). So, (binomial) configurations (of size $\binom{n+1}{2}$) without K_n-graphs are exactly the configurations in which no subspace is a binomial configuration of one-step-smaller size $\binom{n}{2}$.

A binomial $\left(\binom{n+1}{2}_{n-1}\ \binom{n+1}{3}_3\right)$-configuration with two K_n-subgraphs can be considered as an abstract scheme of a perspective of two n-simplices (Prop. 3.6). Generally, the structure of the intersection points of K_n-subgraphs of a $\left(\binom{n+1}{2}_{n-1}\ \binom{n+1}{3}_3\right)$-configuration is isomorphic to a generalized Desargues configuration (Prop. 3.9).

In the paper we do not intend to give a *detailed* analysis of the internal structure of binomial configurations which contain a *prescribed* number of their maximal K_n-subgraphs. We give only a technique (a procedure) to construct a binomial configuration which contains at least the given number of K_n-subgraphs: Theorem 3.12, and we prove that for each admissible integer m there does exist a binomial configuration which freely contains m K_n-subgraphs. Some remarks are also made which show that, surprisingly, binomial configurations with 'many' maximal complete subgraphs (the maximal admissible number, the maximal number reduced by 2, and the maximal number reduced by 3) are the configurations of some well known classes.

2 Definitions

Recall that the term *combinatorial configuration* (or simply a *configuration*) is usually (cf. e.g. [9,22]) used for a finite incidence point-line structure provided that two different points are incident with at most one line and the line size and point rank are constant. Speaking more precisely, a *$(v_r\ b_\varkappa)$-configuration* is a combinatorial configuration with v points and b lines such that there are r lines through each point, and there are $\varkappa$ points on each line. A partial Steiner triple system (a PSTS, in short) is a configuration whose each line contains exactly three points.

In what follows we shall pay special attention to so called *binomial configurations* (more precisely: binomial partial Steiner triple systems, BSTS, in short) i.e. to $\left(\binom{n}{2}_{n-2}\ \binom{n}{3}_3\right)$-configurations with arbitrary integer $n \geq 2$. The class of the configurations with these parameters will be denoted by B_n.

Let k be a positive integer and X a set; we write $\wp_k(X)$ for the set of all k-subsets of X. The incidence structure

$$\mathbf{G}_k(X) := \langle \wp_k(X), \wp_{k+1}(X), \subset \rangle$$

will be called *a combinatorial Grassmannian* (cf. [19]). If $|X| = n$ then $\mathbf{G}_2(X)$ is a B_n-configuration, so it is a BSTS. As the structure $\mathbf{G}_2(X)$ is (up to an isomorphism) uniquely determined by the cardinality $|X|$ in what follows we frequently write $\mathbf{G}_2(|X|)$ instead of $\mathbf{G}_2(X)$. Recall that $\mathbf{G}_2(4)$ is the Veblen (the Pasch) configuration and $\mathbf{G}_2(5)$ is the Desargues Configuration (see e.g. [17,19]). Generally, every structure $\mathbf{G}_2(n)$, $n \geq 5$ will be called a *generalized Desargues configuration* (cf. [4,5,21]).

Let X be a nonempty set with $|X| = n$. A nondirected loopless graph defined on X (we say simply: a *graph*) is a structure of the form $\langle X, \mathcal{P}\rangle$ with $\mathcal{P} \subset \wp_2(X)$. We write $K_X = \langle X, \wp_2(X)\rangle$ for the complete graph with the vertices X; the term K_n is used for an arbitrary graph K_X where $|X| = n$.

We say that a configuration $\mathfrak{N} = \langle X, \mathcal{G}\rangle$ *is contained in* a configuration $\mathfrak{M} = \langle S, \mathcal{L}\rangle$ if

- $X \subset S$.
- each line $L \in \mathcal{G}$ extends (uniquely) to some $\overline{L} \in \mathcal{L}$, and
- distinct lines of $\mathfrak{N}$ extend to distinct lines of $\mathfrak{M}$.

$\mathfrak{N}$ is *l-closed* (is a *subspace* of $\mathfrak{M}$) if each line of $\mathfrak{M}$ that crosses X in at least two points is an extension of a line in $\mathcal{G}$. In that case we also say that $\mathfrak{N}$ is *regularly* contained in $\mathfrak{M}$. Frequently, with a slight abuse of language, we refer to lines of the form $\overline{L}$ ($L \in \mathcal{G}$) as to *sides* of $\mathfrak{N}$.
$\mathfrak{N}$ is *p-closed* if its sides do no intersect outside $\mathfrak{N}$ i.e. when the following holds:

$$L_1, L_2 \in \mathcal{G},\ L_1 \neq L_2,\ p \in \overline{L_1} \cap \overline{L_2} \implies p \in X$$

A graph contained and p-closed in $\mathfrak{M}$ is said to be *freely contained in* $\mathfrak{M}$. In what follows the phrases

X is a K_n-graph (freely) contained in $\mathfrak{M}$ and
K_X is (freely) contained in $\mathfrak{M}$

will be used interchangeably, together with their (admissible) stylistic variants.

3 Complete subgraphs freely contained in PSTS's

Clearly, there are configurations with freely contained subgraphs.

Proposition 3.1 (cf. [12] or [18]) *Let $n \geq 2$ be an integer.* $\mathbf{G}_2(n+1)$ *is a B_{n+1}-configuration which freely contains K_n.*

Proposition 3.2 *Let $n \geq 2$ be an integer. A smallest* PSTS *that freely contains the complete graph K_n is a B_{n+1}-configuration. Consequently, it is a* BSTS.

Proof Let $\mathfrak{M} = \langle S, \mathcal{L}\rangle$ freely contain $K_X = \langle X, \wp_2(X)\rangle$, $|X| = n$. Then $X \subset S$ and for each edge $e \in \wp_2(X)$ there is a third point $\infty_e \in \overline{e}\backslash e$ of $\mathfrak{M}$ on $\overline{e}$. Therefore, $|S| \geq |X| + |\wp_2(X)| = \binom{n+1}{2}$. Write $X^\infty = \{e_\infty : e \in \wp_2(X)\}$. The rank of a point $x \in X$ in $\mathfrak{M}$ is at least $n-1$, so the rank of a point $q \in X^\infty$ is at least $n-1$ as well. On the other hand, there passes exactly one side of K_X through a point in X^∞. Let b_0 be the number of lines through a point in X^∞ distinct from the sides of K_X. This number is minimal when the lines in question contain entirely points in X^∞ and then through $q \in X^\infty$ there pass: one side of K_X and lines contained in X^∞. Consequently, $3b_0 \geq |\wp_2(X)|(n-2)$. This yields $b_0 \geq \binom{n}{3}$. Finally, $|\mathcal{L}| \geq |\wp_2(X)| + b_0 = \binom{n+1}{3}$. □

Proposition 3.3 *Let $\mathfrak{M} = \langle S, \mathcal{L}\rangle$ be a minimal* PSTS *which freely contains a complete graph $K_X = \langle X, \wp_2(X)\rangle$ and $|X| = n$. Then* the complement of K_X *i.e. the structure $\langle S\backslash X, \mathcal{L}\backslash\{\overline{e} \colon e \in \wp_2(X)\}\rangle$ is a B_n-configuration and a subspace of $\mathfrak{M}$.*

Conversely, let $\mathfrak{N} = \langle Z, \mathcal{G}\rangle$ be a B_n-configuration regularly contained in $\mathfrak{M}$. Then $S\backslash Z$ yields in $\mathfrak{M}$ a complete K_n-graph freely contained in $\mathfrak{M}$, whose complement is $\mathfrak{N}$.

Proof By 3.2, $\mathfrak{M}$ is a B_{n+1}-configuration. The first statement was proved, in fact, in the proof of 3.2.

Let $\mathfrak{N}$ be a suitable subconfiguration of $\mathfrak{M}$. Set $X = S\backslash Z$. Then $|X| = n$. Through each point $z \in Z$ there passes exactly one line $L_z \in \mathcal{L}\backslash\mathcal{G}$. If $z_1 \neq z_2$ then $L_{z_1} \neq L_{z_2}$, as otherwise $L_{z_1} \in \mathcal{G}$. We have $|\mathcal{L}\backslash\mathcal{G}| = \binom{n+1}{3} - \binom{n}{3} = \binom{n}{2} = |\{L_z \colon z \in Z\}|$ and therefore $\mathcal{L}\backslash\mathcal{G} = \{L_z \colon z \in Z\}$. Each line in $\mathcal{L}\backslash\mathcal{G}$ contains exactly two elements of X; comparing the parameters we get that $\langle X, \mathcal{L}\backslash\mathcal{G}\rangle$ is a complete graph. □

As we learn from the proofs of 3.2 and 3.3 each minimal partial STS that freely contains a complete graph K_n is associated with a labelling (the map $e \mapsto \infty_e$) of the points of a B_n-configuration $\mathfrak{H}$ by the elements of $\wp_2(X)$, $|X| = n$.

Indeed, let $\mathfrak{H} = \langle Z, \mathcal{G}\rangle$, $\mu \colon \wp_2(X) \longrightarrow Z$ be a bijection, and $|X| = n$. Then the configuration

$$K_X +_\mu \mathfrak{H} := \langle X \cup Z, \mathcal{G} \cup \{\{a, b, \mu(\{a, b\})\} \colon \{a, b\} \in \wp_2(X)\}\rangle$$

is a B_{n+1}-configuration freely containing K_X.

In what follows any bijection μ of the points of a configuration onto an arbitrary set will be frequently named a *labelling*.[1]

The above apparatus was fruitfully applied in [15] to the case $n = 4$ (labelling of the Veblen configuration) to classify 10_3-configurations which freely contain K_4. Clearly, this method can be applied to arbitrary n, though even in the next step: $n = 5$ a classification "by hand" of all the labellings of arbitrary $\left(\binom{5}{2}_3\ \binom{5}{3}_3\right)$-configuration by the elements of $\wp_2(X)$ with $|X| = 5$ seems seriously much more complex, if executable. In what follows we shall propose a way that may be applied to arbitrary n and which seems (at least a bit) less involved.

Corollary 3.4 *Let $\mathfrak{M}$ be a B_{n+1}-configuration. $\mathfrak{M}$ freely contains a complete graph K_n iff $\mathfrak{M}$ regularly contains a B_n-subconfiguration.*

3.1 Intersection properties

Proposition 3.5 *Any two distinct complete K_n-graphs freely contained in a B_{n+1}-configuration share exactly one vertex.*

[1] In [12] the term 'improper points' was used instead of 'labelling'. Now we think that this older term should not be used, as it suggests, incorrectly, some connections with a 'parallelism', something to do with 'directions'.

Proof Let $G_1 = \langle X_1, \wp_2(X_1)\rangle$, $G_2 = \langle X_2, \wp_2(X_2)\rangle$ be two K_n graphs freely contained in a B_{n+1}-configuration $\mathfrak{M}$.

First, we note that $X_1 \cap X_2 \neq \emptyset$. Indeed, suppose that $X_1 \cap X_2 = \emptyset$. Then X_2 is freely contained in the complement of X_1. From 3.3, this complement is a B_n-configuration, which contradicts 3.2.

Let $a \in X_1 \cap X_2$. Since the degree of a in K_{X_1}, in $\mathfrak{M}$, and in K_{X_2} is $n-1$, G_1 and G_2 have common sides through a. Assume that $b \in X_1 \cap X_2$ for some $b \neq a$; as previously the sides of G_1 and of G_2 through b coincide. So, consider arbitrary $x \in X_1 \setminus \{a, b\}$. G_1 and G_2 both contain the sides $\overline{\{a, x\}}$ and $\overline{\{x, b\}}$, so $x \in X_2$. Finally, we arrive to $X_1 = X_2$. ◻

For a geometer the situation considered in 3.5 has a clear geometrical meaning: if a is the common vertex of two complete K_n graphs $\langle X_1, \mathcal{E}_1\rangle$, $\langle X_2, \mathcal{E}_2\rangle$ freely contained in a B_{n+1}-configuration $\mathfrak{M}$ then a is the *perspective center* of two K_{n-1}-simplices $A_1 = X_1 \setminus \{a\}$ and $A_2 = X_2 \setminus \{a\}$. This means: there is a bijective correspondence σ_a between the vertices of the simplices such that for every vertex x of the first simplex the corresponding vertex $\sigma_a(x)$ of the latter simplex lies on the line $\overline{a, x}$ through a and x. As we shall see, in this case also an analogue of a *perspective axis* can be found. That is, there is a subspace Z of $\mathfrak{M}$ and a bijective correspondence ζ between the sides of the simplices A_1 and A_2 such that for each side L of the first simplex the corresponding side $\zeta(L)$ of the latter simplex crosses L in a point on Z.

Proposition 3.6 *Let $G_i = \langle X_i, \wp_2(X_i)\rangle$, $i = 1, 2$ be two complete K_n-graphs freely contained in a B_{n+1}-configuration $\mathfrak{M} = \langle S, \mathcal{L}\rangle$, let $p \in X_1 \cap X_2$, and $\mathfrak{N}_i$ with the pointset $Z_i = S \setminus X_{3-i}$ be the complement of G_{3-i} in $\mathfrak{M}$ for $i = 1, 2$ (cf. 3.3).*

(i) *$\mathfrak{N}_i$ freely contains a complete K_{n-1}-graph, for each $i = 1, 2$.*
(ii) *Each side of G_i missing p crosses exactly one side of G_{3-i} and the latter misses p as well. The intersection points of the corresponding sides form the set $Z_1 \cap Z_2$.*

Proof It seen that $X_i \setminus \{p\}$ is a K_{n-1}-graph contained in Z_i and, clearly, it is p-closed. This proves (i).

Let $p \notin e \in \wp_2(X_i)$, then $\overline{e} \setminus e$ is a point u_e in S. Since $u_e \notin X_i$, we have $u_e \in Z_{3-i}$. Moreover, $u_e \notin X_{3-i}$, since $X_{3-i} \subset \bigcup\{\overline{p, x} : x \in X_i\}$, so $u_e \in Z_i$. Therefore, there is a side e' of G_{3-i} (a line of $\mathfrak{N}_{3-i}$) which passes through u_e. Statement (ii) is now evident. ◻

Even in the smallest possible case ($n = 4$) we have a B_{n+1}-configuration (the fez configuration, cf. [15]) which contains a pair of perspective triangles with the perspective center p such that the correspondence of the form $\overline{\{a, b\}} \longmapsto \overline{\{\sigma_p(a), \sigma_p(b)\}}$ does not yield any perspective axis, but the triangles in question do have a perspective axis.

Proposition 3.7 *Let $\langle X_i, \wp_2(X_i)\rangle$, $i = 1, 2, 3$ be three distinct K_n graphs freely contained in a B_{n+1}-configuration $\mathfrak{M}$. Let $c_k \in X_i \cap X_j$ for all $\{k, i, j\} = \{1, 2, 3\}$. Then $\{c_1, c_2, c_3\}$ is a line of $\mathfrak{M}$.*

Proof We have $c_3 \in X_1 \cap X_2$, so $X_2 \setminus \{c_3\}$ consists of all the ‘third points’ on sides of X_1 through c_3 i.e.

$$X_2 \setminus \{c_3\} = \{\overline{c_3, x} \setminus \{c_3, x\} : x \in X_1 \setminus \{c_3\}\}.$$

From the assumption, $c_1 \in X_2$ and thus $\{c_1, c_3, u\}$ is a line of $\mathfrak{M}$ for some $u \in X_1 \backslash \{c_3\}$. With analogous reasoning we have

$$X_3 \backslash \{c_1\} = \{\overline{c_1, x} \backslash \{c_1, x\} \colon x \in X_2 \backslash \{c_1\}\},$$

so $u \in X_3$. Finally, with 3.5 we have $u = c_2$: the claim. □

Corollary 3.8 *Let $\mathfrak{M} = \langle S, \mathcal{L}\rangle$ be a B_{n+1}-configuration. Let $X_1, X_2, X_3 \in \wp_n(S)$ be pairwise distinct, $\mathcal{E}_i = \wp_2(X_i)$ and $G_i = \langle X_i, \mathcal{E}_i\rangle$ for $i = 1, 2, 3$.*

(i) *Assume that G_1 and G_2 are freely contained in $\mathfrak{M}$. Then G_1, G_2 have exactly $n-1$ common sides i.e.*

$$|\{\overline{e} \colon e \in \mathcal{E}_1\} \cap \{\overline{e} \colon e \in \mathcal{E}_2\}| = n - 1.$$

(ii) *Assume that G_1, G_2, G_3 are freely contained in $\mathfrak{M}$. Then there is exactly one side common to G_1, G_2, and G_3.*

3.2 The structure of complete subgraphs

Next, we pass to an analysis of possible 'many' subgraphs freely contained in a binomial configuration.

Proposition 3.9 *Let $G_i = \langle X_i, \wp_2(X_i)\rangle$, $i = 1, \ldots, m$ be a family of m distinct K_n-graphs freely contained in a B_{n+1}-configuration $\mathfrak{M} = \langle S, \mathcal{L}\rangle$.*

(i) *Set $I = \{1, \ldots, m\}$. The map $q \colon \wp_2(I) \longrightarrow S$ determined (cf. 3.5) by the condition*

$$q^{i,j} = q(\{i, j\}) \in X_i \cap X_j \ \textit{for each}\ \{i, j\} \in \wp_2(I)$$

embeds $\mathbf{G}_2(I)$ into $\mathfrak{M}$.

(ii) *Consequently, $m \leq n + 1$.*

(iii) *If $m = n$ then $\mathfrak{M}$ freely contains one more, $(n+1)$-st K_n-graph.*

Proof Ad (i): From 3.7, $X_i \cap X_j \cap X_k = \emptyset$ for distinct i, j, k in I and thus the map q is an injection. Moreover, 3.7 also yields that q maps each line of $\mathbf{G}_2(I)$ onto a line of $\mathfrak{M}$.
(ii) is immediate now.
Ad (iii): For each $i \in I$ there is exactly one point $d_i \in X_i \backslash \bigcup_{j \neq i} X_j$. Write $X_0 = \{d_i : i \in I\}$, clearly, $|X_0| = n$. Let $i, j \in I$ be distinct; from definition, $q^{i,j} \neq d_i, d_j$. From 3.7 we get that for every $k \in I$, $k \neq i, j$ the side $\overline{q^{i,j}, q^{i,k}}$ of G_i crosses X_j in the point $q^{j,k}$. So, the side $\overline{q^{i,j}, d_i}$ of G_i crosses X_j in a point distinct from all the $q^{j,k}$ i.e. in the point d_j. Thus X_0 is a complete graph with the sides $\{d_i, d_j, q^{i,j}\}$, $\{i, j\} \in \wp_2(I)$. It is seen that X_0 is freely contained in $\mathfrak{M}$. □

Corollary 3.10 *A B_{n+1}-configuration $\mathfrak{M}$ freely contains $n+1$ K_n-graphs iff $\mathfrak{M} \cong \mathbf{G}_2(n+1)$.*

Proof In view of 3.9(i) it suffices to note that the sets $S(i) = \{e \in \wp_2(I) \colon i \in e\}$ are the maximal cliques of $\mathbf{G}_2(I)$ which are not lines of $\mathbf{G}_2(I)$ (cf. [19]). It is seen that each of them is freely contained in $\mathbf{G}_2(I)$ for arbitrary set I with $|I| \geq 3$. □

The results obtained can be summarized in the following Proposition.

Proposition 3.11 *Let $G_i = \langle X_i, \wp_2(X_i)\rangle$, $i = 1, \dots, m$ be a family of m distinct K_n-graphs freely contained in a B_{n+1}-configuration $\mathfrak{M} = \langle S, \mathcal{L}\rangle$. Set $I = \{1, \dots, m\}$, $Z_i := X_i \setminus \bigcup_{k \in I\setminus\{i\}} X_k$, $Z := S \setminus \bigcup_{i \in I} X_i$, $\mathcal{E}_i := \{\overline{e} \colon e \in \wp_2(X_i)\}$, $\mathcal{G}_i := \mathcal{E}_i \setminus \bigcup_{k \in I\setminus\{i\}} \mathcal{E}_k$, $\mathcal{G} := \mathcal{L} \setminus \bigcup_{i \in I} \mathcal{E}_i$ for every $i \in I$, $q^{i,j} \in X_i, X_j$, $Q := \{q^{i,j} \colon \{i,j\} \in \wp_2(I)\}$.*

- (i) *If $L \in \mathcal{G}$ then $L \subset Z$.*
- (ii) *Let $L \in \mathcal{L}$. If $|L \cap Z| \geq 2$ then $L \in \mathcal{G}$.*
- (iii) *$|Z_i| = n - m + 1$ for every $i \in I$.*
- (iv) *Let $\{i,j\} \in \wp_2(I)$. Then $Z_i \cup \{q^{i,j}\}$ and $Z_j \cup \{q^{i,j}\}$ are two K_{n-m+2}-graphs with the common sides through $q^{i,j}$. Each of them is freely contained in $\mathfrak{M}$.*
- (v) *$|Z| = \binom{n+1-m}{2}$*
- (vi) *$|\mathcal{G}| = \binom{n+1-m}{3}$*
- (vii) *Let $L \in \mathcal{G}_i$ for an $i \in I$. Then $|L \cap X_i| = 2$ and $L \cap X_i \subset Z_i$, $|L \cap Z| = 1$*
- (viii) *Let $e \in \wp_2(Z_i)$ for an $i \in I$. Then $\overline{e} \in \mathcal{G}_i$.*
- (ix) *$|\mathcal{G}_i| = \binom{|Z_i|}{2} = \binom{n+1-m}{2}$ for every $i \in I$.*
- (x) *Let $i \in I$. Through every point $p \in Z$ there passes exactly one $L \in \mathcal{G}_i$.*
- (xi) *The structure $\langle Z, \mathcal{G}\rangle$ is a B_{n-m+1}-configuration regularly contained in $\mathfrak{M}$.*

Proof (i): Suppose $a \in L \cap X_i$ for some $i \in I$ and $L \in \mathcal{G}$. Comparing point-ranks we note that all the lines of $\mathfrak{M}$ through a are the sides of G_i, so $L \in \mathcal{E}_i$: a contradiction.
(ii): Suppose $L \notin \mathcal{G}$, then there are $i \in I$ and an edge e of G_i such that $L = \overline{e}$. Clearly, $|L \cap X_i| = 2$, so $|L \cap Z| \leq 1$.
(iii): Evident: $Z_i = \{x \in X_i \colon x \neq q^{i,k}, k \in I\setminus\{i\}\}$ and $|I\setminus\{i\}| = m - 1$.
(iv): Evidently, any two points in Z_i and any two points in Z_j are on a line of $\mathfrak{M}$: a suitable side of G_i or of G_j resp. Let $a \in Z_i$. Then $\overline{q^{i,j}, a} \setminus \{q^{i,j}, a\}$ is a point b of X_j. Suppose $b = q^{j,k}$ for some $k \in I$. From 3.7, $a = q^{i,k}$: a contradiction; thus $b \in Z_j$.
(v): By 3.5 and 3.7, $|\bigcup_{i \in I} X_i| = m \cdot n - \binom{m}{2} \cdot 1 =: \gamma(n)$; we compute $\binom{n+1}{2} - \left(\binom{m+1-n}{2} + \gamma(n)\right) = 0$.
(vi): Analogously, by 3.7 and 3.8, $|\bigcup_{i \in I} \mathcal{E}_i| = m \cdot \binom{n}{2} - \binom{m}{2} \cdot (n-1) + \binom{m}{3} \cdot 1 =: \delta(n)$, and then $\binom{n+1}{3} - (\binom{n+1-m}{3} + \delta(n)) = 0$.
(vii): It is clear that any $L \in \mathcal{G}_i \subset \mathcal{E}_i$ crosses X_i in a pair a, b of points. Suppose $a \notin Z_i$. Then $a = q^{i,k}$ for some $k \in I$, $k \neq i$ and then $L \in \mathcal{E}_i, \mathcal{E}_k$. This yields $a, b \in X_i$. Suppose $L = \{a, b, c\}$ and $c \in X_k$ for $k \in I$. Then $L \in \mathcal{E}_k$, $k \neq i$, so $L \notin \mathcal{G}_i$.
(viii): Suppose $\overline{e} \notin \mathcal{G}_i$, so there is $k \neq i$ such that $\overline{e} \in \mathcal{E}_k$. This means: G_k contains an edge e' with $\overline{e} = \overline{e'}$. Then $e \cap e' \neq \emptyset$, so $e \cap X_k \neq \emptyset$, which contradicts $e \subset Z_i$.

(ix): Immediately follows form (vii) and (viii).
(x): In view of (vii), the map $\mathcal{G}_i \ni L \longmapsto p \in L \cap Z$ is well defined. Clearly, it is injective. From (ix) and (iii) it is also surjective, and this is exactly the claim.
(xi): Immediate, after (i), (ii), (v), and (vi). □

Let $I = \{1, \ldots, m\}$ be arbitrary, let $n > m$ be an integer, and let X be a set with $n-m+1$ elements. Let us fix an arbitrary B_{n-m+1}-configuration $\mathfrak{B} = \langle Z, \mathcal{G}\rangle$. Assume that we have two maps μ, ξ defined: $\mu\colon I \longrightarrow Z^{\wp_2(X)}$ and $\xi\colon I \times I \longrightarrow S_X$, such that $\xi_{i,i} = \mathrm{id}$, $\xi_{i,j} = \xi_{j,i}^{-1}$, and μ_i is a bijection for all $i, j \in I$. Let $S = Z \cup (X \times I) \cup \wp_2(I)$ (to avoid silly errors we assume that the given three sets are pairwise disjoint). On S we define the following family $\mathcal{L}$ of blocks

$$\begin{aligned}\mathcal{L} = \mathcal{G}&\\ &\cup \text{ the lines of } \mathbf{G}_2(I)\\ &\cup \big\{\{\{i,j\},(x,i),(\xi_{i,j}(x),j)\}\colon \{i,j\}\in\wp_2(I), x\in X\big\}\\ &\cup \{\{(a,i),(b,i),\mu_i(\{a,b\})\}\colon \{a,b\}\in\wp_2(X), i\in I\}.\end{aligned}$$

Write

$$m \bowtie_{\xi}^{\mu} \mathfrak{B} = \langle S, \mathcal{L}\rangle. \tag{1}$$

It needs only a straightforward (though quite tidy) verification to prove that

$$\mathfrak{M} := m \bowtie_{\xi}^{\mu} \mathfrak{B} \textit{ is a } B_{n+1}\textit{-configuration}.$$

For each $i \in I$ we set $Z_i = X \times \{i\}$, $S_i = \{e \in \wp_2(I)\colon i \in e\}$, and $X_i = Z_i \cup S_i$. It is seen that $\mathfrak{M}$ *freely contains* m K_n*-graphs* $X_1, \ldots, X_m$. Indeed, let us write $a \oplus b = c$ when $\{a, b, c\}$ is a line (of the configuration in question). Then we have $\{i, j\} \oplus \{i, k\} = \{j, k\}$, $(a, i) \oplus (b, i) = \mu_i(\{a, b\})$, and $(a, i) \oplus \{i, j\} = (\xi_{i,j}(a), j)$. It is seen that the point $\{i, j\}$ is the perspective center of two subgraphs Z_i, Z_j of $\mathfrak{M}$. So, we call the configuration $m \bowtie_{\xi}^{\mu} \mathfrak{B}$ *a system of perspective* $(n-m+1)$*-simplices*. Define $\mu_i^o\colon \wp_2(Z_i) \longrightarrow Z$ by the formula $\mu_i^o(\{(x, i), (y, i)\}) = \mu(\{x, y\})$; it is seen that the configuration $\mathfrak{B}$ is the common 'axis' of the configurations $\langle Z_i, \wp_2(Z_i)\rangle +_{\mu_i^o} \mathfrak{B}$ contained in $\mathfrak{M}$.

Note that the words 'perspective', 'axis', and 'simplices' are used to suggest some *formal* similarities to objects considered in geometry. Such a usage does not mean that the considered binomial configuration $m \bowtie^m u_\xi \mathfrak{B}$ is necessarily realizable in a desarguesian projective space.

Let us consider two special cases of the above definition of a system of perspective simplices.

(i) Let $m = n-1$. Then $\mathfrak{B}$ consists of a single point p: the *center* of $\mathfrak{M}$. Consequently, μ_i is constant, $\mu_i \equiv p$. Moreover, the set X which appears in the definition has 2 elements and then $|S_X| = 2$. Then instead of a map ξ one can consider a graph $\mathcal{P} \subset \wp_2(I)$ defined by $\{i, j\} \in \mathcal{P} \iff \xi_{i,j} = \mathrm{id}$. And then *the system* $m \bowtie_{\xi}^{p} \langle\{p\}, \emptyset\rangle$ *of* m *perspective segments (of 2-simplices) is the multiveblen configuration* $\mathbb{W}^m \rhd_{\mathcal{P}} \mathbf{G}_2(m)$ (cf. [18,20]).

(ii) Let $m = n - 2$. Then $\mathfrak{B}$ consists of a single 3-point line, $L = \{a, b, c\}$. Up to a permutation of X there is a unique bijection $\mu\colon \wp_2(X) \longrightarrow L$. Finally *the system of m perspective triangles* $m \bowtie_{\xi}^{\mu} \langle L, \{L\}\rangle$ *is the system of triangle perspectives* $\mathbf{P}_I \triangleright_{\xi} \mathbf{G}_2(I)$ (cf. [14]).

Now, till the end of this section writing 'a configuration contains m graphs' we mean 'a configuration contains *at least m* graphs'.

Theorem 3.12 *Let $\mathfrak{M}$ be a B_{n+1}-configuration. The following conditions are equivalent.*

(i) *$\mathfrak{M}$ freely contains m K_n-graphs.*
(ii) *$\mathfrak{M}$ is a system of m perspective $(n - m + 1)$-simplices i.e. $\mathfrak{M} \cong m \bowtie_{\xi}^{\mu} \mathfrak{B}$ for a B_{n-m+1}-configuration $\mathfrak{B}$ and some (admissible) maps μ, ξ.*

Proof We have already noticed that $m \bowtie_{\xi}^{\mu} \mathfrak{B}$ contains required subgraphs.

Let $\mathfrak{M} = \langle S, \mathcal{L}\rangle$ and let $X_1, \ldots, X_m \in \wp_n(S)$ be pairwise distinct. Assume that $G_i = \langle X_i, \wp_2(X_i)\rangle$ is freely contained in $\mathfrak{M}$ for every $i = 1, \ldots, m$. Let us adopt the notation of 3.11.

Set $\mathfrak{B} = \langle Z, \mathcal{G}\rangle$. Let us fix a $(n - m + 1)$-element set X and let $\nu_i : Z_i \longrightarrow X$ be a fixed bijection for each $i \in I$. Let $a, b, \in X$ and $i, j \in I$. Define

$$\text{if } a \neq b \quad \text{then } \mu_i(\{a, b\}) = \overline{\nu_i(a)\nu_i(b)} \setminus \{\nu_i(a), \nu_i(b)\},$$

$x_{i,i} = \mathrm{id}_X$, and

$$\text{if } i \neq j \quad \text{then } \xi_{i,j}(a) = b \text{ iff } \{q^{i,j}, \nu_i(a), \nu_j(b)\} \in \mathcal{L}.$$

Finally, we define on the points of $\mathfrak{M}$ the following map F:

$$F\colon \begin{cases} Q \ni q^{i,j} & \longmapsto \{i, j\} \\ Z_i \ni x & \longmapsto (x, i) \\ Z \ni a & \longmapsto a \end{cases}$$

It is a standard student's exercise to compute that F is an isomorphism of $\mathfrak{M}$ and $m \bowtie_{\xi}^{\mu} \mathfrak{B}$. □

Corollary 3.13 *Let $\mathfrak{M}$ be a B_{n+1}-configuration.*

(i) *$\mathfrak{M}$ freely contains $n - 1$ graphs K_n iff $\mathfrak{M}$ is (isomorphic to) a multiveblen configuration.*
(ii) *$\mathfrak{M}$ freely contains $n - 2$ graphs K_n iff $\mathfrak{M}$ is (isomorphic to) a system of triangle perspectives.*

Particular instances of 3.10 and 3.13 in case $n = 4$ can be found in [15]: a 10_3 configuration contains four K_4 iff it contains five K_4 iff it is a Desargues Configuration; a 10_3 configuration contains three K_4 iff it is a multiveblen configuration i.e. iff it is the Desargues or it is the Kantor 10_3G-configuration (cf. [11]); a 10_3 configuration contains two K_4 iff it is a system of triangle perspectives i.e. it is one of the following: the Desargues, the Kantor 10_3G, or the fez configuration.

4 Existence problems

Proposition 4.1 *If there is a B_n-configuration which freely contains exactly m maximal complete subgraphs where $m \leq n-2$ then there exists a B_{n+1}-configuration which freely contains exactly $m+1$ maximal complete subgraphs.*

Proof Let $Y_1, \dots, Y_m$ be the K_{n-1}-subgraphs of a B_n-configuration $\mathfrak{M} = \langle S, \mathcal{L}\rangle$. Set $I = \{1, \dots, m\}$. Let us reprezent $\mathfrak{M}$ as a system of perspectives, so let $q_{i,j} \in Y_i \cap Y_j$ for distinct i, j and $Q = \{q_{i,j} \colon \{i, j\} \in \wp_2(I)\}$, $G_i = Y_i \setminus Q$, and let Z be an "axis" i.e. the intersection of all the complementary subconfigurations of the Y_i'es. Let X be an arbitrary set disjoint with S of cardinality n and let $P \in \wp_m(X)$. Let us number the elements of P: $P = \{p_1, \dots, p_m\}$ and the elements of $X \setminus P$: $X \setminus P = \{x_1, \dots, x_{n-m}\}$. Note that $n - m \geq 2$. For each $\{i, j\} \in \wp_2(I)$ we introduce the triple $\{p_i, p_j, q_{i,j}\}$ as a new line. The number of points in each of the sets G_i is $n - m$; let σ_i be an arbitrary bijection of G_i onto $X \setminus P$. The second family of new lines consists of the triples $\{p_i, x, \sigma_i(x)\}$ with $x \in G_i$, $i \in I$. Finally, the third class of the new lines consists of the triples $\{x_i, x_j, \mu(x_i, x_j)\}$, where μ is an arbtrary labelling of the edges of the graph $K_{X\setminus P}$ by the elements of Z: it is possible due to cardinalities of the sets in question. Let $\mathfrak{M}^*$ be the structure defined on the point universe $S \cup X$, whose lines are the lines of $\mathfrak{M}$ and the three classes of new lines introduced above. It is seen that $\mathfrak{M}^*$ is a B_{n+1}-configuration. It is also evident that K_X and $K_{Y_i \cup \{p_i\}}$ for $i \in I$ are K_n-subgraphs freely contained in $\mathfrak{M}^*$. Suppose that $\mathfrak{M}^*$ contains another freely contained K_n-subgraph K_Y, let $p \in X \cap Y$. Then $Y_0 := Y \setminus X = Y \setminus \{p\}$ is a K_{n-1} subgraph of $\mathfrak{M}$. Consequently, $Y_0 = Y_i$ for some $i \in I$. Suppose that $p \neq p_i$, then the two subgraphs $K_{Y_i \cup \{p_i\}}$ and K_Y freely contained in $\mathfrak{M}^*$ have more than a point in common. Consequently, $p = p_i$ and $Y = Y_i \cup \{p_i\}$. Thus $\mathfrak{M}^*$ freely contains exactly $m + 1$ K_n-subgraphs. □

Proposition 4.2 *If there exists a B_{n+1}-configuration which freely contains exactly two K_n graphs then there is also a B_{n+1}-configuration without any K_n-subgraph freely contained in it.*

Proof Let $n \geq 4$. Let $\mathfrak{M} = \langle S, \mathcal{L}\rangle$ be a B_{n+1}-configuration which freely contains exactly two complete K_n-graphs X_1, X_2. Let $p \in X_1 \cap X_2$. Set $A_i = X_i \setminus \{p\}$. Let $e_1 \in \wp_2(A_1)$ and $e_2 \in \wp_2(A_2)$ such that $\overline{e_1} \cap \overline{e_2} \ni q$ for a point q (cf. 3.6(ii)) Let $e_1 = \{a_1, b_1\}$, $e_2 = \{a_2, b_2\}$ such that $b_2 \notin \overline{p, a_1}$ and $a_2 \notin \overline{p, b_1}$. We replace the two lines $\{q, a_1, b_1\}$ and $\{q, a_2, b_2\}$ of $\mathfrak{M}$ by two other triples $\{q, a_1, b_2\}$ and $\{q, b_1, a_2\}$; let $\mathfrak{M}^*$ be the obtained incidence structure. Clearly, $\mathfrak{M}^*$ is a B_{n+1}-configuration. $\mathfrak{M}^*$ *does not freely contain any* K_n-graph

Indeed: Suppose that $\mathfrak{M}^*$ freely contains a K_n-graph Y. Let us have a look at the collinearity graph $A_{\mathfrak{K}}$ of an arbitrary configuration $\mathfrak{K}$. Clearly, if K_X is contained in $\mathfrak{K}$ then K_X is a subgraph of $A_{\mathfrak{K}}$. In our case exactly two edges e_1, e_2 of $A_{\mathfrak{M}}$ were replaced by two (other) edges $\{a_1, b_2\}, \{a_2, b_1\}$ to form $A_{\mathfrak{M}^*}$. So, K_Y cannot be build entirely from the edges missing $e_1 \cup e_2$.

(1) $Y \cap e_1 \neq \emptyset \neq Y \cap e_2$. Indeed, suppose, eg. $Y \cap e_1 = \emptyset$. The same pairs of points in $S \setminus e_1$ (except e_2) are collinear in $\mathfrak{M}$ and in $\mathfrak{M}^*$ and therefore Y is a K_n-graph in $\mathfrak{M}$: a contradiction.

(2) $e_1 \not\subset Y$ and $e_2 \not\subset Y$: the pair of points in e_1 is not collinear in $\mathfrak{M}^*$, and, analogously the points in e_2 are also not collinear.

So, Y contains exactly one point x_1 in e_1 and one point y_2 in e_2. Clearly, x_1, y_2 must be collinear in $\mathfrak{M}^*$.

(3) Suppose that $y_2 \in \overline{p, x_1}$. Without loss of generality we can take $x_1 = a_1$, $y_2 = a_2$. Then $b_1, b_2 \notin Y$. For points in $S \setminus \{b_1, b_2\}$ exactly the same pairs are collinear in $\mathfrak{M}$ and in $\mathfrak{M}^*$, so Y is a subgraph of $\mathfrak{M}$, which is impossible.

Without loss of generality we can assume that $a_1, b_2 \in Y$ and $a_2, b_1 \notin Y$. The following three cases should be considered:

(4) $a_2 \in \overline{p, a_1}$, and $b_2 \in \overline{p, b_1}$,
(5) $a_2 \in \overline{p, a_1}$ and $b_2 \notin \overline{p, b_1}$,
(6) $a_2 \notin \overline{p, a_1}$ and $b_2 \notin \overline{p, b_1}$.

(4): Note that the sides of Y and the lines of $\mathfrak{M}^*$ through vertices of Y coincide. So, $\{p, a_1\}$ is an edge of Y and thus $p \in Y$. Take any point $c_1 \in A_1 \setminus e_1$. Then c_1, b_2 are not collinear in $\mathfrak{M}^*$, so $c_1 \notin Y$. Let $c_2 \in \overline{p, c_1} \setminus \{p, c_1\}$ (this line of $\mathfrak{M}$ was unchanged), then $c_2 \in Y$. But c_2 and a_1 are not collinear in $\mathfrak{M}^*$ and a contradiction arizes.

(5): In this case also necessarily $p \in Y$. Let $c_2 \in \overline{p, b_1} \setminus \{p, b_1\}$; then $c_2 \neq b_2$ and $c_2 \in Y$. But, contradictory, a_1, c_2 are not collinear in $\mathfrak{M}^*$.

(6): Now, either $p \in Y$ or $c_2 \in Y$, where $\{p, a_1, c_2\} \in \mathcal{L}$ ($c_2 \in X_2$). If $p \in Y$ then we take $c_1 \in \overline{p, a_2} \setminus \{p, a_2\}$; then $c_1 \in Y$. An inconsistency appears, as c_1, b_2 are not collinear in $\mathfrak{M}^*$. Consequently, $c_2 \in Y$ and $p \notin Y$. Therefore, the point c_1 in $\overline{p, b_2} \setminus \{p, b_2\}$ is in Y ($c_1 \in X_1$). But c_1, c_2 are not collinear in $\mathfrak{M}^*$, though. □

As an important consequence we obtain now

Theorem 4.3 *Let m, n be integers, $4 \leq n$, and $1 \leq m \leq n-1$ or $m = n+1$. Then there exists a B_{n+1}-configuration which freely contains exactly m K_n-graphs.*

Proof Let $n \geq 4$ be an integer and let $J(n)$ be the set of integers m such that there is a B_{n+1}- configuration with exactly m freely contained subgraphs K_n. From 3.9, $J(n) \subseteq \{0, 1, \ldots, n-1, n+1\} =: F(n)$. We need to prove that $J(n) = F(n)$ for every integer $n \geq 4$. Clearly, this equality holds for $n = 4$ (cf. [15]).

Assume that $J(n) = F(n)$ holds for an integer n.

From 4.1 and 3.10, $J(n+1) \supseteq \{1, 2, \ldots, n, n+2\}$. Then from 4.2 we get $0 \in J(n+1)$ and therefore $J(n+1) = F(n+1)$. By induction, we are done. □

Remark 4.4 There is no reasonable B_{0+2}-configuration, there is exactly one B_{0+3}-configuration: a line $\mathbf{G}_2(3)$, with exactly 3 freely contained copies of K_2, and there is exactly one B_{0+4}-configuration: the Veblen configuration $\mathbf{G}_2(4)$, which freely contains 4 copies of K_3.

5 Other known examples

5.1 Combinatorial Veronesians

Let us adopt the notation of [17]. Let $|X| = 3$. Then the combinatorial Veronesian $\mathbf{V}_k(X)$ is a B_{k+2}-configuration; its point set is the set $\mathfrak{y}_k(X)$ of the k-element multisets with elements in X. The maximal cliques of $\mathbf{V}_k(X)$ were established in [8]. From that results we learn that

Fact 5.1 *The K_{k+1} graphs freely contained in $\mathbf{V}_k(X)$ are the sets $X_{a,b} := \mathfrak{y}_k(\{a,b\})$, $X_{b,c} := \mathfrak{y}_k(\{b,c\})$, and $X_{c,a} := \mathfrak{y}_k(\{c,a\})$. Its axis is the set X^k. For $u \in \wp_2(X)$ the B_{k+1}-subconfiguration of $\mathbf{V}_k(X)$ complementary to X_u (with the universe $y\mathfrak{y}_{k-1}(X)$, $y \in X \setminus u$) is isomorphic to $\mathbf{V}_{k-1}(X)$.*

Corollary 5.2 *A B_{k+1}-Veronesian with $k > 2$ freely contains exactly three complete K_k-graphs.*

5.2 Quasi Grassmannians

Let us adopt the notation of [21]. The configuration $\mathfrak{R}_n$ is a B_{n+2}-configuration. Recall the role of the set $X = \{1, 2\}$ if n is even and $X = \{0, 1, 2\}$ if n is odd. Namely, let us cite after [21] the following

Fact 5.3 *The complete K_{n+1}-graphs freely contained in $\mathfrak{R}_n$ are the sets $S(i) = \{a \in \wp_2(Y) \colon i \in a\}$, where $\wp_2(Y)$ is the point set of $\mathfrak{R}_n$, $X \subset Y$, and $i \in X$.*

Corollary 5.4 *Let $\mathfrak{M}$ be a B_n-quasi-Grassmannian, $n > 2$. If n is even then $\mathfrak{M}$ freely contains exactly two K_{n-1}-graphs, and it freely contains exactly three K_{n-1}-graphs when n is odd.*

Besides, the above indicates one more similarity between combinatorial Veronesians and quasi Grassmannians represented as a fan of configurations 10_3G.

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